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THE HOSSFELD FUNCTION:
An Alternative Model for Depicting Stand Growth and Yield

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Abstract

This paper (1) sets out two forms of a sigmoid growth equation called the Hossfeld function, (2) examines its basic properties and (3) assesses its utility for modelling growth and yield of even-aged forest stands. A small data set comprising several re-measurements of mean top height and net basal area/ha in sixty Douglas fir permanent sample plots in the Central North Island of New Zealand was used to test the fit of the Hossfeld function to real data. The fit of mean top height on age was slightly better for the polymorphic than for the anamorphic form. When compared with the polymorphic log-reciprocal and anamorphic Chapman-Richards, the polymorphic Hossfeld function for mean top height produced a slightly more precise fit and equally satisfactory residual patterns. Similarly, net basal area/ha was equally well modelled over time by the Hossfeld, log-reciprocal and modified Gompertz functions. The Hossfeld function, therefore, should be recognized by researchers as an equation well worth considering in the development of compatible growth and yield models.

Introduction

Over the past twenty-five years, forest scientists have adopted a variety of sigmoid-shaped functions to model growth and yield of forest stands through time. Clutter (1963), for example, utilized the log-reciprocal equation suggested by Schumacher (1939), to model basal-area development of thinned *Pinus taeda* L. This model proved to be most useful for forest growth modelling, and has been used extensively with many species [see, for example, Bennett (1970b) Leak *et al.* (1970), Perala (1971), and Woollons and Hayward (1985)]. Occasionally, however, some scientists have chosen alternative yield functions. Pienaar and Turnbull (1973) utilized the Chapman-Richards equation to predict basal area of *Pinus elliottii* Engelm from South African

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CCT permanent sample plots [O'Connor (1935)], while Nokoe (1978) used a modification of the Gompertz yield function to model volume per hectare in three species (*Thuja plicata, Pinus contorta,* and *Pseudotsuga menziesii*). Then again, Yang *et al.* (1978) advocated the use of the Weibull function to produce flexible growth curves, while Smith and Kozak (1984) extolled the utility of the Schnute (1981) model. Leech and Ferguson (1981) review the performance of several yield models for radiata pine in Australia.

Researchers in Japan have been much concerned with theoretical aspects of growth and yield modelling of unthinned or self-thinned stands. For example, Minowa (1982, 1983), Naito (1984) and Sweda (1984) have reviewed various optional forms of characterizing whole stand approaches while Tanaka (1987) has concentrated on a diameter class technique. The emphasis on theory in Japanese studies appears to be largely a consequence of having insufficient field data to test the theory. Theoretical and practical considerations need to interact, however, to gain a useful perspective. This paper illustrates, therefore, the importance of such interactions on desirable forms of growth models.

All these sigmoid functions mentioned above are linked in that each can be derived from linear first-order differential equations. But Umemura (1984) argued, again theoretically, the case for growth models based on second order differential equations, and developed two yield functions as a result of considering distinct or coincident roots. This concept is certainly worth exploring further; attention here is focussed meanwhile on the use of only first-order differential equations. A later contribution will be directed at practical evaluation of Umemura's hypothesis.

In this contribution, we study an equation not known extensively in Western countries, namely the Hossfeld function [Peschel (1938) and Prodan (1968)]. Some of its statistical properties are developed, and its utility is demonstrated by fitting it to a forest dataset. The results are discussed, and the utility of the function in relation to some commonly employed others is reviewed.

Hossfeld Equation : Statistical Properties

The Hossfeld function is given by

$$Y = (\alpha T^{\gamma})/(\alpha\beta + T^{\gamma}) \tag{1}$$

where in (1)

Y = yield at time T

α, β, γ = parameters (to be estimated)

Evidently, when $T = 0$, then $y = 0$, and when $T \to \infty$, then

$$Y = \lim_{T \to \infty} (\alpha T^{\gamma})/(\alpha\beta + T^{\gamma})$$

$$= \lim_{T \to \infty} \alpha\gamma T^{(\gamma-1)}/(\gamma T^{(\gamma-1)})$$

$$= \alpha \text{ [by L'Hopitals rule]}$$

Differentiating (1) with respect to T gives the *growth* equation

$$dY/dT = \alpha\beta\gamma Y/(T(\alpha\beta+T^{\gamma})) \qquad (2)$$

Differentiating (2) again, to give the second differential, d^2Y/dT^2, and setting the resultant expression to zero, produces a point of inflexion:

$$Y_{inflex} = \alpha(\gamma\text{-}1)/2\gamma \qquad \text{at } T = (\alpha\beta(\alpha-1)/(\alpha + 1))^{(1/\gamma)} \qquad (3)$$

Thus, equation (1) is shown to be a sigmoid growth curve, which goes through the origin, with an upper asymptote, α. From (2), we note that growth is postulated to be a function of current yield, Y, inverse of age, T, and a maturation factor, $(\alpha\beta + T\gamma)^{-1}$. The yield equation, therefore, makes good biological as well as mathematical sense.

Equations such as (1) can be enhanced by expressing them as difference equations [Clutter (1963); Clutter *et al.* (1983), Borders *et al.* (1984)]. For a nominated yield, Y_1, at time T_1. Equation (1) can then be re-expressed as:

$$Y_2 = f(Y_1, T_1, T_2 \mid \alpha,\beta) \qquad (4)$$

Two forms of (4) are available, *anamorphic* or *polymorphic* ([Borders *et al.* (1984)]. With the latter, the shape parameter β is postulated to be site specific, and eliminated from equation (4) in the following manner:

$$Y_1 = \alpha T_1^{\gamma}/(\alpha\beta_i + T_1^{\gamma}) \text{ and } Y_2 = \alpha T_2/(\alpha\beta_i^{\gamma} + T_2^{\gamma}) \quad (5)$$

Eliminating βi from (5) leads to the difference equation

$$(1/Y2) = (T_1/T_2)^{\gamma}(1/Y_1) + (1/\alpha)\ [1-(T_1/T_2)^{\gamma}]$$

or (6)

$$Y_2 = 1/((T_1/T_2)^{\gamma}(1/Y_1) + (1/\alpha)\ [1-(T_1/T_2)^{\gamma}])$$

Alternatively an anamorphic form of (4) can be constructed by regarding the asymptote α to be site specific, which gives the difference equation:

$$(1/Y_2) = (1/Y_1) + \theta(1/T_2^{\beta}-1/T_1^{\beta})$$

or (7)

$$Y_2 = 1/((1/Y_1) + \theta(1/T_2^{\beta}-1/T_1^{\beta}))$$

where in (7)

θ = a function of β and α

Methods

Plantation forests in New Zealand are dominated by the conifer *Pinus radiata*, but there are also significant holdings of, for example, Eucalypts and Douglas-fir [*Pseudotsuga menziesii*]. A Douglas fir data set was chosen to test the utility of the Hossfeld projection equations for both mean top height and net basal area/ha developed above. Mean top height estimates, defined as the average height of the largest 100 stems/ha by diameter, and net basal area/ha were available from 60 permanent sample plots. A summary of relevant plot statistics is given in Table 1.

Table 1: Summary of Plot Data

	Mean	Min.	Max.	
Age (measured)	35	13	61	(years)
Top height	27.7	10.6	43.9	(m)
Site index*	31.7	19.5	38.8	(m)
Plot stocking	952	158	2411	(stems/ha)
Net basal area	59.9	15.9	110.1	(m^2/ha)

* defined as plot mean top height, at age 40.

A plot of top-height development against stand age for all data is given in Figure 1.

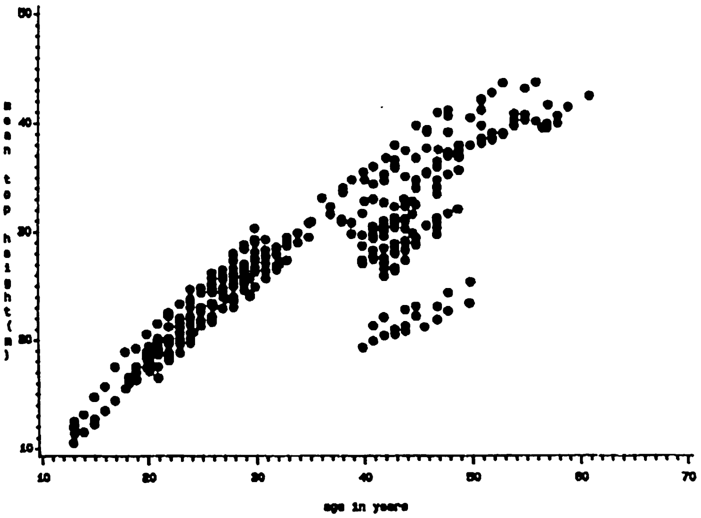

Figure 1: Mean Top Height on Age for Douglas-fir Data Set

Equations (6) and (7) were then fitted to the relevant data using the NLIN procedure of the SAS statistical system (SAS Inst [1985]).

Results and Discussion

Parameter estimates for the two models of mean top height were:

Polymorphic form (equation 6)

α = 0.014 733 s.e. = ±0.001 013

γ = 1.489 727 s.e. = ±0.074 146

Residual mean square, σ^2 = 0.336 04

A plot of residual data values against predicted figures [equation 6] is given in Figure 2; the data are evidently well-behaved, with no sign of bias or systematic patterns, indicative of an excellent goodness-of-fit (Draper and Smith [1981], Chapter 3).

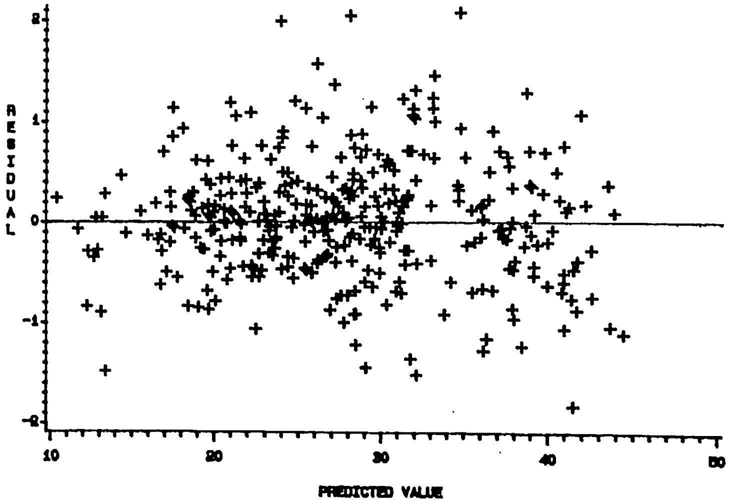

Figure 2: Residual Patterns for Mean Top Height Against Predicted Values

Anamorphic form (equation 7)

θ = 1.910 298 s.e. = ±0.234 505

β = 1.244 341 s.e. = ±0.048 215

Residual mean square, σ^2 = 0.375 06

A plot of residual against predicted values [not given here] gives a generally random appearance, but less satisfactorily so than in Figure 2 above.

It is thus clear that both projection equations, (6) and (7) give excellent models of top height development, but equation (6), the polymorphic form, is somewhat superior. This can be substantiated by comparing the respective residual mean squares, where that associated with equation 6, is 10% lower in value. Nevertheless, either form has provided a satisfactory fit.

The projection form (6) can be transformed moreover, to generate a set of site-index curves; by definition, site index, S, is that mean top height which is achieved at age 40. By substituting $T_2 = 40$, and S (site index) = $\bar{h}_{100,2}$ we have

$$1/S = 1/\bar{h}_{100,1}\,(T_1/40)^{\gamma} + (1/\alpha)\,(1-(T_1/40)^{\gamma})$$

or (8)

$$(1/\bar{h}_{100,1}) = [(1/S)-(1/\alpha)\,(1-(T_1/40)^{\gamma})]\,(T_1/40)^{-\gamma}$$

Figure 3 shows a set of site-index curves, resultant from equation (8); convergence to a common asymptote, indicative of a polymorphic function, is clearly depicted.

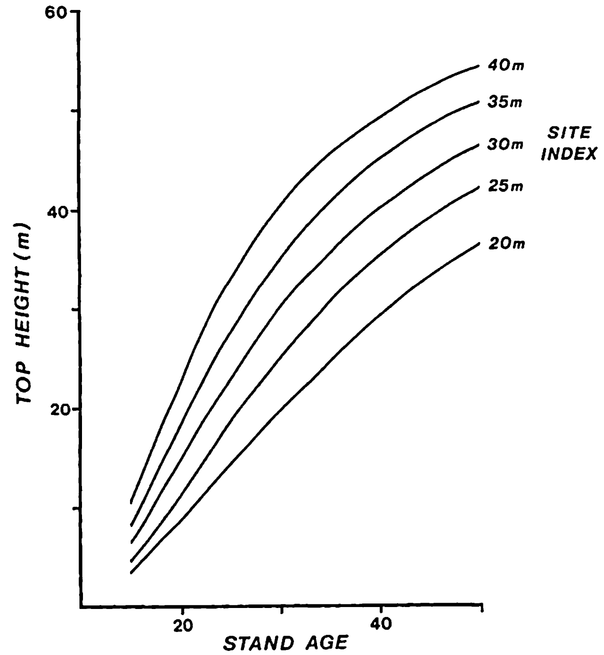

Figure 3: Polymorphic Site Index Curves for Douglas-fir Data Set

In the above analysis, no tabulation of analyses of variance, or t-statistics (associated with parameters) is provided because of serial-correlation existing in the data, obtained from repeated measurements of identical sets of trees; as such, error terms will be underestimated [Davis and West (1981), Ferguson and Leech (1981)]. Nevertheless, this complication has no significant effect in practice; careful use of graphical plots as assayed here allows goodness-of-fit assessment, while in any event the t-statistics are usually irrelevant, since the majority of parameters tested are established by a *priori* knowledge [Woollons and Hayward (1985)].

A reasonable fit of net basal area/ha for this data set was also achieved with equation 7. The estimated coefficients, α = 118.6875 ± 2.3650 γ = 2.045 73 ± 0.057 77, and σ^2 = 1.3189 were precise enough: residual patterns, moreover, showed a reasonably satisfactory spread, with every indication that improvements to the fit could be made through the addition of the usual kinds of other explanatory variables, such as stocking and thinning index, [see Weir (1989)].

The Hossfeld function is thus shown here to be an excellent model for depicting top height development and basal area/ha with the data at hand. At no time, however, do we claim it to be an optimal equation for modelling stand growth in general; alternative models may well give equal precision, or the function may not perform as well with alternative species or data.

Two other frequently used equations were fitted to the mean top height data for comparative purposes.

(1) The log reciprocal yield equation (Schumacher, 1939),

$$Y = \exp(\alpha + \beta/T^{\gamma}) \tag{9}$$

can be modified to the following projection form,

$$\bar{h}_{100,2} = \bar{h}_{100,1}^{(T_1/T_2)^{\gamma}} \exp[\alpha(1-(T_1/T_2)^{\gamma})] \tag{10}$$

(2) The Chapman-Richards has the yield form (Clutter *et al.*, 1984)

$$Y = \alpha(1 - \exp(-\beta T))^{1/(1-\gamma)} \tag{11}$$

and a corresponding projection form,

$$\bar{h}_{100,2} = \bar{h}_{100,1}((1 - \exp(-\beta T_1))/(1 - \exp(-\beta T_2)))^{1/(1-\gamma)} \tag{12}$$

The term "exp" represents exponential function

The fitted coefficients, their standard errors and the model residual mean square are shown in Table 2.

Table 2: Comparison of Hossfeld, Log-reciprocal and Chapman-Richards Equations for Estimating Mean Top Height

Function	α	β	γ	σ^2
Hossfeld	0.014 73 ± 0.001 01	—	1.489 72 ± 0.07415	0.336 04
Log-reciprocal	5.113 ± 0.24002	—	0.477 59 ± 0.06209	0.350 02
Chapman-Richards	—	0.029 64 ± 0.003 08	1.493 41 ± 0.087 39	0.337 13

Residual patterns for the Schumacher and Chapman-Richards equations are similar to those for the Hossfeld function. The Hossfeld function, however, provides a marginally better fit than either of the other two to this particular set of mean top height data.

Its utility is not confined simply to modelling height growth, however. Zeide (1988 *pers. comm.*) has, for example, found that it provides a good fit to diameter growth of individual trees. We have compared here its performance in predicting net basal area/ha growth with the log-reciprocal and Gompertz functions. The yield and projection forms for the last one are respectively:

$$Y = \exp(\alpha \exp(-\gamma T - {}^{1}/_{2}\, \delta T^2) \qquad (13)$$

$$G_2 = \exp(\log G_1 \exp(-\gamma (T_2 - T_1) + \delta (T_2^2 - T_1^2)) + \alpha (1 - \exp(-\gamma (T_2 - T_1) + \delta (T_2^2 - T_1^2)))) \qquad (14)$$

The fits for all three models do not differ to any great extent, as can be seen from Table 3.

Table 3: Comparison of Hossfeld, Log-reciprocal and Gompertz Functions for Estimating Net Basal Area/ha.

Function	α	β	γ	δ	σ^2
Hossfeld	118.6875 ± 2.3649	2.0457 ± 0.0578	—	—	1.318
Log-reciprocal	5.0208 ± 0.0394	0.9989 ± 0.03441	—	—	1.2736
Gompertz	4.9406 ± 0.0438	—	0.08485 ± 0.00389	0.000707 ± 0.000038	1.2873

The log-reciprocal equation performs slightly better than either of the other two, but all three functions are amenable to the addition of further predictor variables and other types of formulation (see Weir, 1989) which would undoubtedly sharpen up their precision. As it is, the standard error of estimate for basal area is less than 2 m^2/ha for all three, a most acceptable level of precision. As for mean top height, use of the Hossfeld function for modelling net basal area/ha should be seriously considered.

We believe it is unrealistic to expect a unique sigmoid function to consistently perform better than others with forest growth and yield data. A more rational approach is to be aware of the existence of several candidate equations, and to explore their utility with data in question. The Hossfeld function provides one viable alternative that should be considered.

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